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Numerical Studies of New Stellarator Concepts

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A three-dimensional computer code has been developed to study the magnetohydrodynamic equilibrium and stability of a diffuse or sharp boundary plasma in toroidal geometry. It is shown how equilibria with net toroidal current identically zero can be determined and how growth rates of instabilities can be calculated. Applications are made to an $l = 2, 3$ stellarator configuration that offers the possibility of achieving a critical value as high as 10% for the plasma parameter β.

1. INTRODUCTION

We have recently published a book [2] describing a magnetohydrodynamics code to compute equilibrium and stability for a plasma in toroidal geometry with significant three-dimensional distortions. The principal merit of the code is its ability to resolve three-dimensional effects, but because of this it is rather expensive to run. The question arises whether physical applications can be found that justify such expensive calculations. The model that the code is best suited to analyze is the classical stellarator. In this paper we shall present sample calculations that are relevant to some modern stellarator concepts. The physical interest of the results is enhanced by recent progress in neutral beam heating which offers the prospect of achieving practical stellarator equilibria in the laboratory.

Classical stellarators with simple $l = 2$ or $l = 3$ windings do not turn out to provide adequate containment when the plasma parameter β, defined to be the ratio between the maximum fluid pressure p and the maximum magnetic pressure $B^2/2$, significantly exceeds 2 or 3%. Because the Maxwell stress tensor is quadratic in the components of the magnetic field B, and because comparable quadratic terms appear in the stellarator expansion of Greene and Johnson [3], the addition theorem

$$2\cos[l\theta - kz]\cos[(l+1)\theta - kz] = \cos\theta + \cos[(2l+1)\theta - 2kz]$$

for the cosine suggests that a restoring force to balance toroidal drift can be obtained by suitably combining l and $l + 1$ windings. This twists the magnetic lines nearest the principal axis of the torus so that their lengths more evenly match those at the outer circumference, which can be chosen to be circular. Such ideas have been used to define high-β stellarator equilibria [4]. Unfortunately the large aspect ratios and

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many periods required to attain really high values of β lead to so-called $m = 1$ instabilities. These cannot be avoided even by introducing triangular cross sections of a kind proposed in some of our earlier publications [2]. However, we shall establish that satisfactory intermediate configurations with aspect ratios A on the order of 10 and with β as large as 10% can be found for stellarators with combined $l = 2$ and $l = 3$ windings.

Our computer code is the best theoretical tool available to study the $l = 2, 3$ stellarator concept. We have developed a new version that defines equilibria with net current I equal to zero on every toroidal flux surface. This requirement is related to diffusion phenomena that can be modeled by the code, and it eliminates some of the resistive instabilities that plague the Tokamak program. To make a convincing case for theorems affirming stability of magnetohydrodynamic modes for the $l = 2, 3$ stellarator, we have refined our method of determining equilibria and calculating growth rates. In the discussion of these improvements of the theory, some familiarity with our book [2] and with the notation used there will be assumed.

In the next section we present a similarity solution of the partial differential equations defining magnetohydrodynamic equilibrium that is helpful in validating the computer code. It furnishes a check on three-dimensional effects observed in calculations of flux surfaces with triangular cross sections. More precisely, for high β the triangularity has not been found to decay inside the plasma as rapidly as had been expected from potential theory. This observation is confirmed for a long wavelength, helically symmetric equilibrium solution where the cross sections all have the same shape and for which $\beta = 1$ at the magnetic axis. The solution also serves as a prime example with which to assess the accuracy of numerical estimates of growth rates.

The analysis of stability by numerical methods is strongly influenced by truncation errors. Some of these take the form of additional constraints due to discretization, and they have a stabilizing effect comparable to that of finite Larmor radius. Others behave more like artificial resistivity due to numerical relaxation of flux constraints, but their destabilizing effect can be significantly greater than that of the physical resistivity. In practice we have found that to test stability in a reliable way with our code it is necessary to calculate growth rates of the discrete model with extreme accuracy and then extrapolate carefully to the limit of zero mesh size. In this paper we shall present a new procedure for estimating discrete growth rates which is related to the classical Rayleigh–Ritz method of computing eigenvalues. More specifically, we represent growth rates in terms of shifts in the potential energy by formulas quite analogous to those appearing in the standard variational principle of magnetohydrodynamics [3]. This approach turns out to have a decisive advantage over estimates based on Fourier analysis of time-dependent processes.

In the final section of the paper the question of which configuration might lead to the largest critical value of β is considered subject to the requirement of zero net current. The $l = 2, 3$ stellarator seems to be the best model we know of at this time. Comparisons are made with experimental data, notably with those that are available for the Proto–Cleo stellarator [5], in order to substantiate conclusions based on the theory.

2. Magnetic Flux Surfaces with Triangular Cross Sections

Our computations of the toroidal equilibrium of a high-β plasma whose flux surfaces have triangular cross sections exhibit remarkable penetration of the triangularity into the plasma. Motivated by this observation, we seek similarity solutions of the equilibrium equations such that all cross sections have the same shape. Helically symmetric equilibria that we shall obtain in this fashion serve to check the accuracy of the calculations. The information they provide about growth rates is especially important because that is the area where the most troublesome questions about resolution arise.

Our point of departure is the partial differential equation

$$\chi_{xx} + \chi_{yy} = a\chi^{\alpha},$$

which describes a family of straight cylindrical equilibria in two dimensions. Contrary to usual practice, we shall allow the exponent α to be fractional. Separating variables in polar coordinates r and θ and putting $\alpha = (n - 2)/n$ and $a = n^2$, we find that there are solutions of the form

$$\chi = r^n[1 + f(\theta)],$$

where f satisfies the ordinary differential equation

$$f'' + n^2(1 + f) = n^2(1 + f)^{(n-2)/n}.$$

This equation has periodic solutions with the period 2π for appropriate choices of the eigenvalue parameter n. In particular, if the amplitude of f is small, the equation can be linearized so that it reduces to

$$f'' + l^2 f = 0$$

with $l^2 = 2n$. Here the periodic solutions are $\cos l\theta$ and $\sin l\theta$ with integral l.

The case $l = n = 2$ is classical and corresponds to flux surfaces with elliptical cross sections. More interesting from our point of view is the case $l = 3$, $n = 9/2$, which yields flux surfaces $\chi =$ const. that all have similar triangular cross sections. For this example, which has no physical singularity at the origin, we have obtained excellent agreement with runs of the computer code. It follows that the coordinate system on which the code is based gives adequate resolution at the magnetic axis.

To arrive at comparable similarity solutions with more complicated geometry in three dimensions, let us consider the partial differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{r\chi_r}{1 + k^2r^2} + \frac{1}{r^2}\chi_{\theta\theta} + \frac{2B_*}{(1 + k^2r^2)^2} + \frac{1}{k^2}\left[p' + \frac{B_*B_*'}{1 + k^2r^2}\right] = 0$$

for helically symmetric equilibria (cf. [3]). Here $\chi = \chi(r, \theta - kz)$ is a flux function of the cylindrical coordinates r, θ, and z, while $p = p(\chi)$ is the pressure and $B_* = B_*(\chi)$ is the component of the magnetic field parallel to the lines of helical symmetry.

For a later application we note that the equation has vacuum field solutions of the form

$$\chi(r, \theta) = g(r) + h(r) \cos l\theta$$

that define nested families of cylindrical flux surfaces between which lie islands surrounded by a separatrix of higher topological structure. Such examples indicate that it is naive to assume the existence of nested toroidal flux surfaces sweeping out a vacuum magnetic field that is bounded by perfectly conducting tori. However, such an assumption can be justified for discrete models whose resolution is anyhow inadequate to describe finer structure of the solution. This problem will be discussed in more detail in another publication.

At present we are concerned with the long wavelength model obtained by neglecting terms in k^2r^2. In that limiting case equilibria can be found by setting

$$p + \tfrac{1}{2}B_*^2 = \text{const.}$$

and solving the reduced partial differential equation

$$\chi_{rr} + \frac{1}{r}\chi_r + \frac{1}{r^2}\chi_{\theta\theta} = -2B_* + r^2 B_* B_*' .$$

By putting

$$B_* = -n\chi^{(n-2)/n}$$

and separating variables, we see that there are similarity solutions of the same form

$$\chi = r^n[1 + f(\theta - kz)]$$

as before, but with f satisfying the new ordinary differential equation

$$f'' + n^2(1 + f) = 2n(1 + f)^{(n-2)/n} + n(n - 2)(1 + f)^{(n-4)/n}.$$

Again we consider the linearized version of the eigenvalue problem defining periodic solutions. Now the integer l associated with the eigenfunctions is related to the exponent n by the rule

$$4(n - 1) = l^2.$$

The simplest choice $l = n = 2$ corresponds to constant pressure and zero current. It leads to a classical $l = 2$ stellarator field that is useful for validation of the computer code in three dimensions. However, we shall concentrate here on the less familiar case $l = 3$, $n = 13/4$, which provides the desired example of an equilibrium with $\beta = 1$ at the magnetic axis and with triangular cross sections whose shapes are all similar.

The plots shown in Fig. 1 represent four different cross sections of magnetic flux surfaces from one run of our computer code for the similarity solution with $l = 3$ and with the amplitude of f approximately equal to 0.5. The similarity of the shapes helps to validate the code in the case of fully three-dimensional calculations.

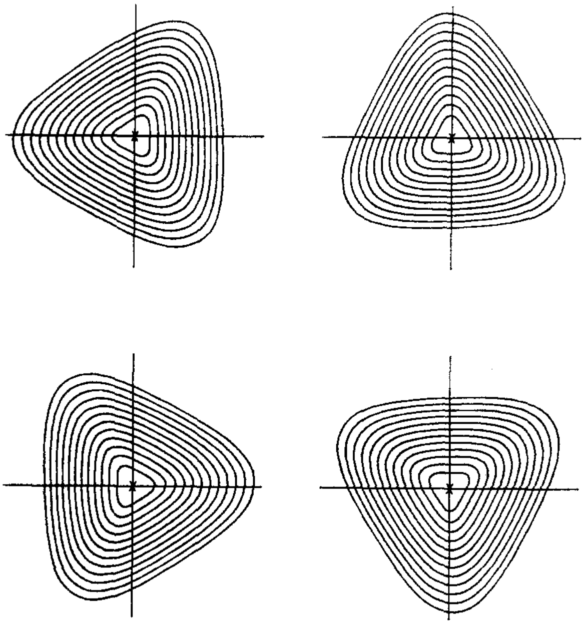

FIG. 1. Cross sections of the similarity solution.

For large β more generally there is a certain resonance of the triangularity that enables it to penetrate the plasma in an unexpected fashion. We have studied solutions of this kind numerically in connection with high-β stellarator applications of the code. To test our conclusions we shall undertake next to examine the stability of the similarity solution.

The classical variational principle of magnetohydrodynamics asserts that an equilibrium is stable if the second variation

$$\delta W = \tfrac{1}{2} \int [| Q + J \times \nu(\xi \cdot \nu)|^2 + \gamma p \mid \nabla \cdot \xi \mid^2 + K \mid \xi \cdot \nu \mid^2] \, d\tau$$

of the potential energy is positive for all nontrivial choices of an infinitesimal displacement ξ of the plasma [3]. Here γ is the adiabatic exponent, ν is the normal to a nested family of flux surfaces, $J = \nabla \times B$ is the current density, $Q = \nabla \times (\xi \times B)$ is the first variation of the magnetic field B, and

$$B^2 K = -\nabla p \cdot \nabla(2p + B^2) - (J \cdot B)^2 + (J \cdot B)(B \times \nu) \cdot \nabla \times (B \times \nu).$$

Growth rates $i\omega$ of unstable modes are defined by extrema of the Rayleigh quotient

$$\omega^2 = \frac{2\delta W}{\int \rho \xi^2 \, d\tau}.$$

Let us represent the magnetic field of the similarity solution as a cross product

$$B = \nabla s \times \nabla \psi$$

of Clebsch potentials, where s is a toroidal flux function characterizing nested flux surfaces $s = \text{const.}$, and ψ is a multiple-valued flux function whose period in the toroidal direction reduces to the rotational transform $\mu = \mu(s)$ of the equilibrium, which is constant. It can be shown that without changing the lead term in δW, an alteration in $p' = p'(\chi)$ of the order $k^2 f^2$ and proportional to $\chi^{(n-2)/n}$ can be introduced which makes either μ or the net current I vanish identically. Thus it suffices to carry out the stability analysis of the similarity solution in the simplest case $\mu = 0$.

It turns out that s is proportional to χ and that ψ differs from $\theta/2\pi$ by a function of $\theta - kz$. The infinitesimal displacement ξ of the plasma satisfies the relation

$$\xi \times B = \delta\psi \, \nabla s - \delta s \, \nabla \psi,$$

where $\delta\psi$ and δs are the first variations of the flux functions ψ and s. To study the $m = 1$ modes we consider special variations given by

$$\delta\psi = a(s) \sin 2\pi\psi, \qquad \delta s = b(s) \cos 2\pi\psi,$$

which are admissible because $\mu = 0$. A lengthy calculation then shows that the second variation of the potential energy has a lead term of the form

$$\delta W = -\frac{4(n-2)(n^2+3n-5)}{n^2} \int k^2 f^2 b^2 \cos^2\theta \, d\tau.$$

Thus when $n > 2$ the similarity solution is unstable to a wide class of $m = 1$ perturbations with more or less arbitrary dependence on the radial coordinate s.

The instability of the long wavelength similarity solution casts doubt on controversial results we announced earlier concerning stability of high-β stellarator equilibria with triangular cross sections [1, 2]. Here we observe that instability prevails even when the plasma is bounded by a perfectly conducting outer flux surface on which the pressure does not vanish. Thus wall stabilization fails in the present case of triangular cross sections with $\beta = 1$ at the magnetic axis. Finally, note that by solving an analytic Cauchy problem we can convert such an outer flux surface into the free boundary of a more general model of the equilibrium.

3. Magnetohydrodynamic Growth Rates

Our computer code is based on a discrete version of the variational principle for the potential energy

$$E = \int \left[\frac{B^2}{2} + \frac{p}{\gamma - 1}\right] d\tau$$

in magnetohydrodynamics [2]. The magnetic field B is represented in terms of two Clebsch flux functions s and ψ,

$$B = \nabla s \times \nabla \psi.$$

The first of these is single valued and defines a nested family of toroidal flux surfaces $s =$ const. The second is given by the formula

$$\psi = -u + \mu(s)v + \lambda(s, u, v),$$

where u and v are poloidal and toroidal coordinates with unit periods and λ is single valued. The pressure p is related to the density ρ by the equation of state $p = \rho^\gamma$, and the mass $M(s)$ inside each flux surface $s =$ const. is prescribed. So also is the rotational transform $\mu = \mu(s)$, which specifies the flux constraints of the problem. Only equilibria with constant pressure on each flux surface $s =$ const. are computed. They are considered to be stable whenever E has a relative minimum.

In order to treat the question of stability adequately from the point of view of numerical analysis, it is necessary to compute growth rates on fixed meshes and then extrapolate their values to zero mesh size. Because we use the variational principle, a convenient procedure to estimate growth rates can be built around the formula of Section 2 involving the Rayleigh quotient. We proceed to describe an algorithm for the numerical implementation of this idea that has been incorporated in our computer code.

Let ψ_0 represent an equilibrium at which $E = E(\psi)$ becomes stationary as a functional of ψ and the other unknowns, and let a test function ψ_{kmn} corresponding to a given mode of perturbation of the flux function ψ_0 be assigned. We consider the problem of minimizing the difference

$$\delta W = E(\psi) - E(\psi_0)$$

subject to the constraint that the scalar product

$$(\psi - \psi_0, \psi_{kmn}) = \int (\psi - \psi_0)\, \psi_{kmn}\, d\tau$$

has a fixed amplitude. The value of the Rayleigh quotient

$$\omega^2 = \frac{E(\psi) - E(\psi_0)}{\int \rho \xi^2\, d\tau}$$

corresponding to the extremal function ψ for this problem defines a growth rate associated with the equilibrium ψ_0 and the test function ψ_{kmn}. The norm in the denominator can be estimated by means of the approximate formula

$$\xi^2 = (\delta\psi \, \nabla s - \delta s \, \nabla\psi)^2 / B^2,$$

which is suggested by the analysis in Section 2.

Our choice of a norm and of a scalar product here are adequate for the study of gross $m = 1$ modes. In this case it also suffices to use for ψ_{kmn} only the lead term in a Fourier expansion with respect to the variables s, u, and v, despite the presence of small sidebands associated with toroidal geometry. However, for higher modes it becomes necessary to refine both the formula for the norm and the choice of ψ_{kmn}.

Because δW scales locally like ξ^2, the values of ω^2 computed above are essentially independent of the amplitude of $(\psi - \psi_0, \psi_{kmn})$ in practice. Like the Rayleigh–Ritz principle, the method works well when ψ_{kmn} is a good approximation to some preferred mode because it then gives an even better approximation to the corresponding growth rate. More reliable estimates of growth rates are obtained than can be found by fitting exponentials to quantities depending on the artificial time parameter that occurs in our highly accelerated iterative scheme for the solution of the minimum problem [2].

An expansion of $\delta\psi = \psi - \psi_0$ in eigenfunctions shows that our procedure leads to negative values of δW predicting instability whenever an equilibrium ψ_0 is unstable for a mode to which the test function ψ_{kmn} is not orthogonal. The accuracy of the estimate of the corresponding growth rate is enhanced by its stationary dependence on the choice of the test function, as can be seen from related examples in linear algebra. However, for a diffuse plasma without sharp boundary our finite-difference scheme always furnishes numerical results that represent lower bounds on the exact growth rates of physical instabilities. Hence it is advisable to compare answers at several mesh sizes if one wishes to conclude that an equilibrium is stable.

The method we have described enables one to calculate a growth rate associated with any test function ψ_{kmn}. The success of the procedure for a specific mode depends on one's ability to select the test function appropriately. For many applications in magnetohydrodynamics the choice of ψ_{kmn} is straightforward. It is even possible to estimate growth rates directly for modes with higher values of the wave numbers k and m in the toroidal and poloidal directions. Moreover, information about nonlinear saturation of unstable modes can be obtained by examining the behavior of ω^2 when the amplitude of $(\psi - \psi_0, \psi_{kmn})$ becomes large.

The Rayleigh quotient effectively scales the Alfven speed out of the formula for the growth rate. This is important because we have avoided solving the full system of time-dependent partial differential equations of magnetohydrodynamics. What matters most in settling questions of stability is the sign of the difference $E(\psi) - E(\psi_0)$. The accuracy of the calculations is found to be just as good for small β as it is for large β, and our code is one of the few theoretical tools that are available in an intermediate range.

From a run on a typical mesh we obtain in practice values of E accurate to seven significant figures. These may result after subtraction in estimates of growth rates that have three significant figures. However, a further loss of significance is usually encountered in passing to the limit of zero mesh size.

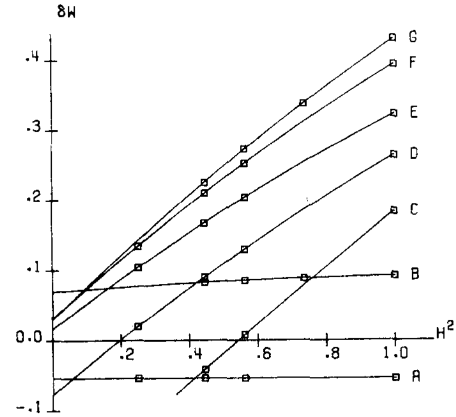

FIG. 2. Growth rates of $m = 1$ modes.

A: $\mu = 0$, $\beta = 0.700$, $\Delta_2 = -0.15$, $\Delta_3 = 0.00$, $\epsilon = 0$;
B: $I = 0$, $\beta = 0.001$, $\Delta_2 = -0.30$, $\Delta_3 = 0.00$, $\epsilon = 0$;
C: $\mu = 0$, $\beta = 1.00$, $\Delta_2 = 0.00$, $\Delta_3 = 0.15$, $\epsilon = 0$;
D: $\mu = 0$, $\beta = 0.700$, $\Delta_2 = 0.00$, $\Delta_3 = 0.15$, $\epsilon = 0$;
E: $I = 0$, $\beta = 0.001$, $\Delta_2 = 0.00$, $\Delta_3 = 0.15$, $\epsilon = 0$;
F: $I = 0$, $\beta = 0.076$, $\Delta_2 = -0.15$, $\Delta_3 = 0.15$, $\epsilon = 0.1$;
G: same as F except $k = 1/13$ for full torus.

The computer code has been used to calculate growth rates of the $m = 1$ mode for the similarity solution introduced in Section 2. Some of the results for δW are plotted against the square of the mesh size h in Fig. 2. The value $h = 1$ is identified with a mesh of 6 intervals in the radial direction and 12 intervals each in the poloidal and toroidal directions. Because the computations turn out to be second order accurate, we have used least squares to plot a parabola through the data for each growth rate in the plane of h^2 and δW. In our labeling of the curves, the parameters Δ_l refer to the shape of a perfectly conducting outer wall whose cross sections are defined by a standard formula

$$re^{i\theta} = [1 - \Delta_0 \cos 2\pi v - \Delta_3 \cos 2\pi(3u - v)]\, e^{2\pi i u} + \Delta_1 e^{2\pi i v} - \Delta_2 e^{-2\pi i(u-v)}$$

from [2].

The data plotted on curve A indicate that for $l = 2$ windings alone it is not necessary to extrapolate to zero mesh size to encounter instability. The truncation error is substantially bigger for the case $l = 3$ of triangular cross sections, but it remains of the order h^2. Curve C represents the example of the similarity solution, which becomes unstable on a mesh of $10 \times 18 \times 18$ points. The error has stabilized all the other runs shown in Fig. 2 that had triangular cross sections. Actually this is a situation ideal for application of the present method because it facilitates computation of the extremal values $E(\psi)$ and $E(\psi_0)$ of the energy that occur in the formula for δW.

The data for curve D were obtained with the same triangularity, the same rotational transform, and the same pressure profile as for the similarity solution, but with a lower value of $\beta = 0.7$ at the magnetic axis. This example was stable on each of the meshes used in the computation, but extrapolation to zero mesh size shows it to be unstable. Such cases have led us in the past to the overly optimistic view that triangular cross sections might stabilize high-β stellarator equilibria [2].

A better understanding of the situation occurring here will follow from a discussion of the results plotted on curve E. In this case an equilibrium with the same triangularity as before, but with relatively small $\beta = 0.001$ and with net current $I = 0$, is shown to be stable to the $m = 1$, $k = 0$ mode by a margin in δW that is narrow but acceptable numerically. Results equivalent to those plotted on curve E are obtained for the Proto-Cleo stellarator, which is stable enough to maintain the plasma in equilibrium for a large fraction of a second [5]. On the other hand, the internal $m = 1$, $k = 0$ mode is found to be neutral from similar runs with $I = 0$ for a pressure profile defined initially by the formula

$$p = 0.35(1 - r^2)^9$$

and used in [2] to model the INTEREX experiment at the Max Planck Institute for Plasma Physics at Garching, which has a compression ratio between 3 and 4 with $\beta = 0.7$. This is not entirely inconsistent with the results presented in [2] for a related but different free-boundary mode, which was found to be stable.

It must be emphasized that accompanying the large triangularity Δ_3 that occurs in these examples there is so much self-inductance of the equilibrium that a significant difference arises between the normalizations $\mu = 0$ and $I = 0$. It is only the cases with zero net current that may be expected to be stable, but they entail such large rotational transforms that stability of the Kruskal-Shafranov kink mode can only be achieved for smaller numbers of periods and smaller aspect ratios than were contemplated in the high-β stellarator program. The outcome of our investigation is that stability of the $m = 1$ mode is primarily controlled by familiar properties of the rotational transform μ, which should remain safely inside the interval $0 < \mu < 1$. Such a restriction is impossible to meet for both a single period and the full torus in cases with very large aspect ratios, which are therefore excluded. Thus stability prevails only for moderate values of β.

It should be noted that in our calculations of growth rates for internal $m = 1$ modes we have found that reasonable accuracy is only achieved by using relatively fine meshes in the direction of the toroidal coordinate v. Some of our earlier work was inadequate in that respect because we attempted to include both $l = 1$ and $l = 3$ windings and to model unnecessarily complicated radial effects all within the scope of one lengthy three-dimensional computation.

4. The $l = 2,3$ Stellarator

We turn our attention to the question of whether three-dimensional geometry can be used to obtain magnetohydrodynamic equilibria with decisively advantageous properties. Tokamak equilibria, which are axially symmetric, suffer from various resistive and ideal magnetohydrodynamic instabilities associated with the net current needed to balance toroidal drift. The corresponding reactor concept has the unfortunate property that it is pulsed. On the other hand, in stellarators the three-dimensional geometry of helical windings is used to compensate for toroidal drift while maintaining zero net current, so that these disadvantages are eliminated.

We shall apply our computer code to search for stellarator equilibria that have as high a critical value of β as possible. To suppress resistive instabilities we impose the requirement $I \equiv 0$ on the net current, and we also ask that the principal $m = 1$ modes of ideal magnetohydrodynamics be stable. Thus steady-state operation can be envisioned.

The condition $I \equiv 0$ is related to problems of resistivity and diffusion. The diffusion equation for the magnetic field B in the case of finite scalar conductivity σ is

$$\sigma B_t = \Delta B,$$

which implies for fixed p that

$$\sigma E_t = \int B \,\Delta B \, d\tau = - \int (\nabla B)^2 \, d\tau < 0$$

if contributions from the boundary are suppressed. Since the physical time scale for this equation is much slower than that of ideal magnetohydrodynamics, it is reasonable to represent the process of diffusion by a one-parameter family of equilibria. The method of steepest descent on which our code is based therefore suggests that we model diffusion crudely by minimizing the energy E with respect to the mass function $M(s)$ and the rotational transform $\mu(s)$, which are the essential input data of the code. Assuming β to be relatively small, or the mass to be in a steady state, we choose to neglect any changes in $M(s)$.

If s itself represents the toroidal flux and $F(s)$ stands for the poloidal flux, so that $\mu = F'(s)$, then an application of the calculus of variations shows that the change in the energy due to a perturbation δF of the poloidal flux is given by the simple formula

$$\delta E = - \int dF \, dI(s) = \int I \, \delta\mu \, ds.$$

We therefore propose to introduce the equation

$$\sigma F_t = I'(s)$$

for a path of steepest descent of E as a primitive model of diffusion. Since I scales like $\mu = F'$, we have here an analog of the heat equation $F_t = F''$. In particular, when the total poloidal flux $F(1) - F(0)$ is fixed, then our model predicts that the net current I will diffuse toward the magnetic axis, raising the level of μ there. Such a phenomenon is observed experimentally and has damaging consequences for the Tokamak concept.

The minimum of E with respect to F is achieved for constant net current $I = I(s)$. In particular, relinquishing the boundary conditions on F and minimizing directly with respect to μ gives $I \equiv 0$. This suggests a simpler iterative scheme to drive I to zero that we can describe in terms of an artificial time parameter t by the equation

$$\mu_t = -I.$$

The most natural way to implement the scheme in our code is by minimizing the discrete approximation to E with respect to the values of μ at mesh points. We have coded such a procedure and have found it to work remarkably well in practice. It provides an improvement of our theory that is essential for the study of stellarators. However, a word of caution is called for if it is proposed to calculate vacuum fields by driving I to zero in a pressureless plasma. The counterexample of Section 2 concerning nested families of flux surfaces indicates that only weak solutions of this problem can be expected to exist, and they may be interspersed with islands and current sheets modeling the higher topological structure of the magnetic lines in the vacuum.

The windings of a classical stellarator are associated with a single poloidal multiplicity l, which usually has the value $l = 2$ or $l = 3$. The effect of the windings is to produce a rotational transform that stabilizes the plasma. This stabilization is enough to compensate for toroidal drift when β is no bigger than 2 or 3 %. However, for

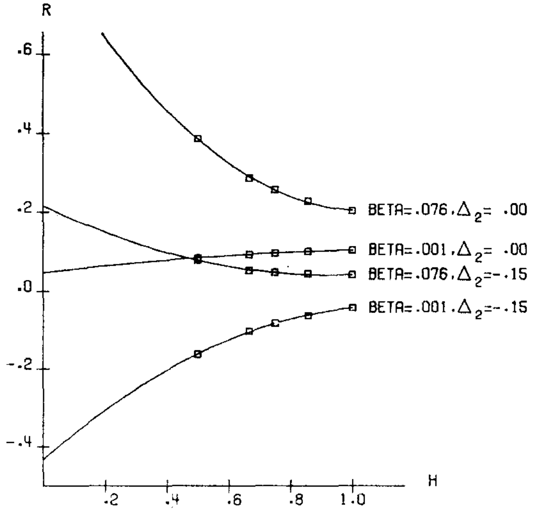

FIG. 3. Equilibria of the $l = 2, 3$ stellarator.

$I = 0$; $\Delta_3 = 0.15$; $\epsilon = 0.1$; $QLZ = 13$.

larger values of β it becomes necessary to introduce another mechanism providing more restoring force so that the plasma will remain adequately centered within the containing coils. This can be achieved by combining l and $l + 1$ windings as indicated in Section 1. In an intermediate range of β, and at lower and more practical aspect and compression ratios, the $l = 0, 1$ combination used in the Scyllac project turns out to be ineffectual. From runs of the computer code it appears that the configuration leading to the highest permissible critical value of β is an $l = 2, 3$ stellarator. The latter configuration has a desirable distribution of rotational transform both for equilibrium and for stability.

In Fig. 3 we present numerical results for a selection of stellarator equilibria that have been calculated subject to the requirement $I \equiv 0$ of no net current. In order to estimate the location of the magnetic axis we have plotted a dimensionless coordinate r as a function of the mesh size h, which is normalized as in Fig. 2. The quantity r, which measures average distance from the center of the coils in units of the wall radius and is oriented to increase in the outward direction, is found to be only first order accurate. Hence to extrapolate to zero mesh size we fit parabolas in the (h, r)-plane by least squares through data from five different grids.

It can be seen that an $l = 3$ winding alone with $\Delta_3 = 0.15$ yields an acceptable equilibrium for $\beta = 0.001$, whereas to contain a plasma with the larger value $\beta = 0.076$ we have had to include both $l = 2$ and $l = 3$ windings with $\Delta_3 = -\Delta_2 = 0.15$. In the case of the latter geometry the magnetic axis moves continuously outward over a physically reasonable range of r as β increases from 0.001 to 0.076. It is remarkable that even in a vacuum the $l = 2, 3$ stellarator field shifts the magnetic axis significantly inward. This can be verified independently by using the sharp boundary version of the code. The initial values of the pressure profile for the computations were of the form

$$p = p_0(1 - r^2)^N$$

with $N = 2$, but the solution was found to be relatively insensitive to the exponent N. The aspect ratio of the outermost flux surface, which we may interpret as a separatrix, was $1/EP = 10$, and there were $QLZ = 13$ periods of the windings.

Curve F in Fig. 2 displays data for the $m = 1$, $k = 0$ growth rate of the $l = 2, 3$ stellarator we have described. Corresponding data for the Kruskal–Shafranov mode of the full torus with 13 periods, which we refer to as the $m = 1$, $k = 1/13$ mode, are presented on curve G. The run on the finest grid for this case, which consisted of $10 \times 18 \times 234$ mesh points, was performed on the CRAY computer at the National Magnetic Fusion Energy Computer Center. It took 2 hr of machine time for 2500 iterations. Extrapolation to zero mesh size shows that the equilibrium is stable to both $m = 1$ modes with approximately equal margins that are safe from the standpoint of numerical errors. For larger choices of k we found δW to be even bigger. These favorable stability results can be attributed to the fact that with $I \equiv 0$ the rotational transform for the full torus came out in the interval $0.4 \leqslant \mu \leqslant 0.8$. Growth rates of equilibria with $I \equiv 0$ are, of course, calculated with μ fixed.

For both equilibrium and stability it is to be observed that on any fixed mesh crude enough to be feasible in practice most of the calculations turn out to be misleading. It is only after appropriate extrapolations to zero mesh size have been performed that physically significant conclusions can be drawn. Unless care is exercised in this matter it is all too easy to fall into the temptation of believing that stable equilibria exist with unrealistically high values of β.

Further calculations have shown that somewhat higher β might be achieved by passing to $l = 2, 3$ stellarators with other choices of Δ_2, Δ_3, EP, and QLZ. However, there is small prospect of reaching critical values of β much above 10 %. For purposes of comparison we have also studied $l = 1, 2$ stellarator configurations. They produce a maximum β of little more than 5 %, but their mathematical theory does have the advantage that the truncation error for $m = 1$ growth rates is considerably reduced. This is reflected in the flat slopes of curves A and B in Fig. 2. Higher choices of the poloidal multiplicity l seems to be inappropriate because the effect of the windings decays too rapidly in both the vacuum and the plasma regions. Perhaps the least attractive possibility is the $l = 0, 1$ stellarator, which has little self-inductance. In this connection we mention that for higher β the Elmo Bumpy Torus (EBT) can be interpreted in our model as an $l = 0$ stellarator with equal but opposite $l = +1$ and $l = -1$ sideband fields whose rotational transforms just cancel each other out. However, important physical phenomena are ignored in such a model.

The apparent success of the $l = 2, 3$ stellarator concept is an outgrowth of our optimistic theory about stability for equilibria with triangular cross sections [2]. To establish more confidence in the physics of our conclusions it is worthwhile to point out that good agreement of the calculations is obtained with experimental data for the Proto–Cleo stellarator [5]. This is shown by curve E in Fig. 2 and the example $\beta = 0.001$ and $\Delta_2 = 0$ in Fig. 3, which serve to model Proto–Cleo data. If the INTEREX experiment at Garching is completed it may shed further light on the situation. Ultimately we hope that an $l = 2, 3$ stellarator experiment will be constructed using our theory and computer code as a guide.

Higher modes, especially those with $m = 2$, can be studied by a refinement of the techniques we have described. It is possible to treat nonlinear saturation of instabilities and bifurcated equilibria, too (cf. [2]). However, ballooning modes have not yet been examined by the method because of limitations on computer capacity for three-dimensional calculations. There is some prospect that they may be less dangerous for the $l = 2, 3$ stellarator anyway because the aspect ratio is large and the plasma is well centered inside the coils.

In closing we mention that we have used the code to make preliminary estimates of the critical β for a Tokamak of aspect ratio four with realistic profiles for the pressure and the rotational transform. A value exceeding 5 % has been obtained, which is in the range of our $l = 2, 3$ stellarator data and of the most recent experimental results.

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